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Solving Difficult Reachability Problems in JuMP.jl

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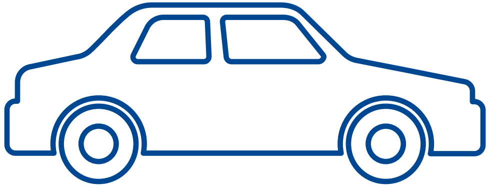
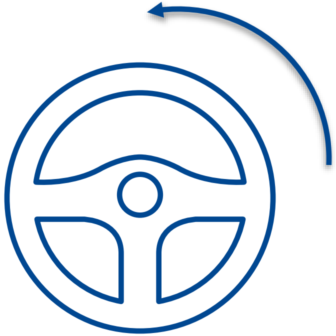
Background: What is a control system?

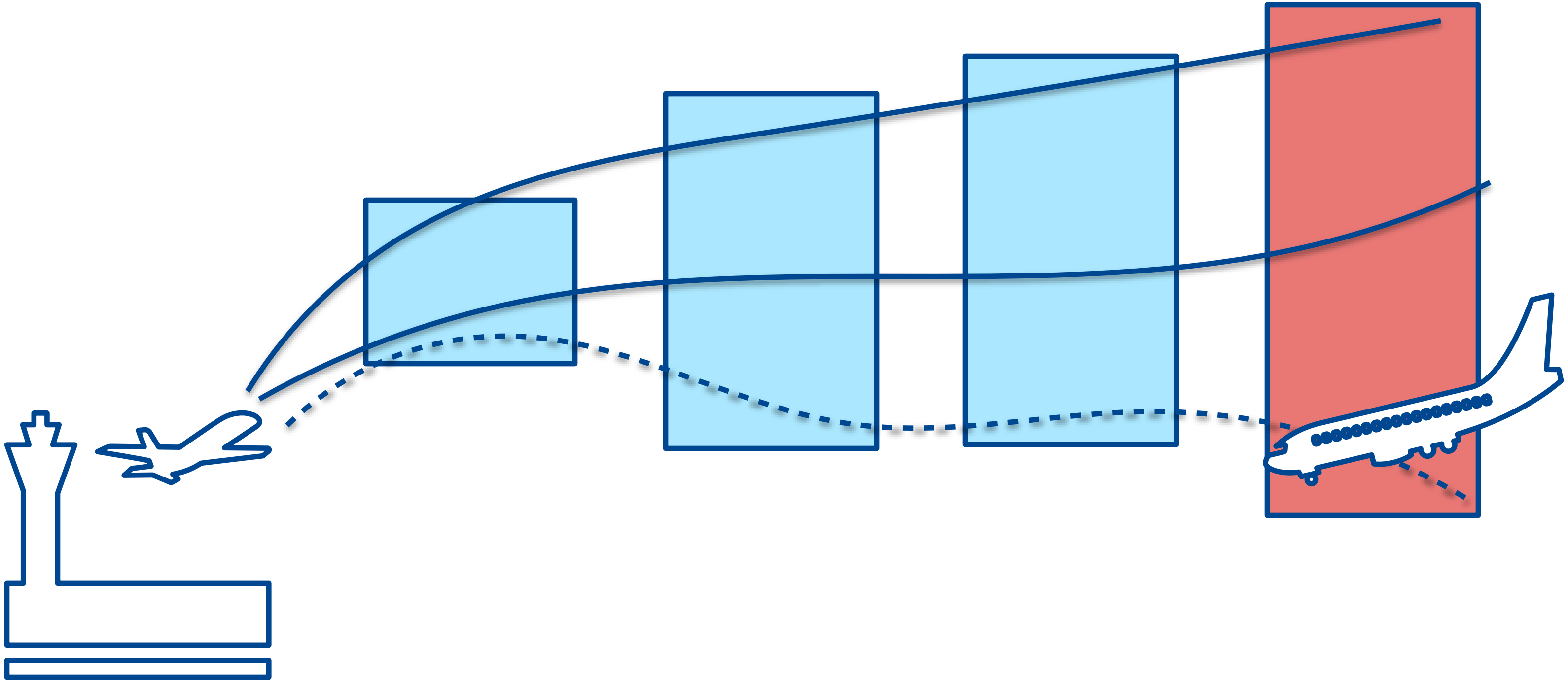


Dynamical system



that takes control inputs
(In this case from a computer, not a human)

Reachability Analysis is Important for Control Systems



Reachability analysis for nonlinear dynamical systems with neural network control policies

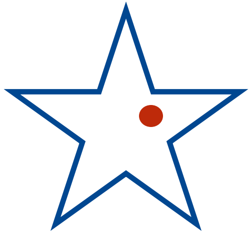
OVERTVerify.jl

https://github.com/sisl/OVERTVerify.jl

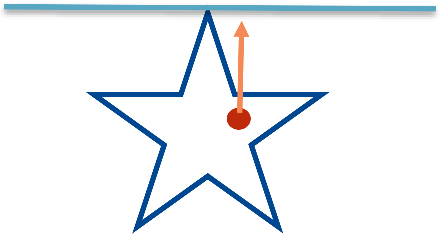
Sidrane, Chelsea, Amir Maleki, Ahmed Irfan, and Mykel J. Kochenderfer. "OVERT: An algorithm for safety verification of neural network control policies for nonlinear systems." *Journal of Machine Learning Research* 23, no. 117 (2022): 1-45.

Optimization To Solve Reachability Problems

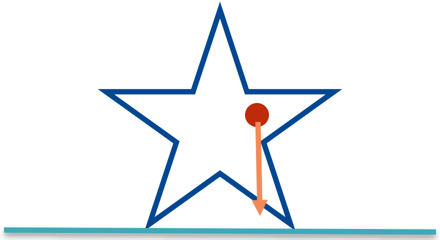
Method 1: Explicit Computation



Encode system constraints and find initial feasible point



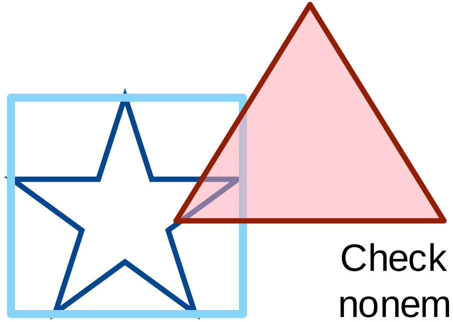
Maximize



Minimize

Optimization To Solve Reachability Problems

Method 1: Explicit Computation, e.g., and an avoid set



Check for nonempty intersection

Compute minimal enclosing hull of reachable set with optimizer

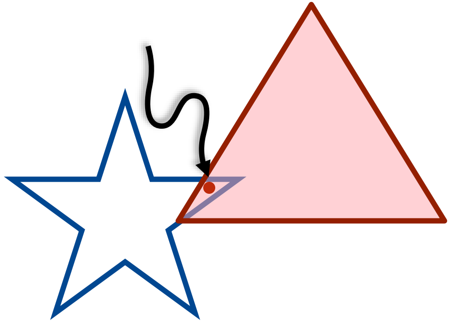
Ability to change objective in JuMP.jl while keeping constraints is helpful

LazySets.jl, of course

Repeat for every timestep

Optimization To Solve Reachability Problems

Method 2: Feasibility Check

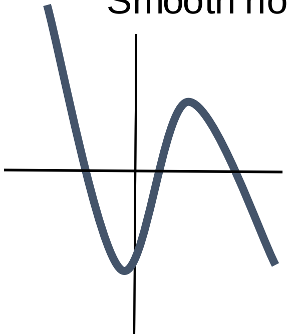


Encode system constraints and avoid set into optimizer; find feasible point

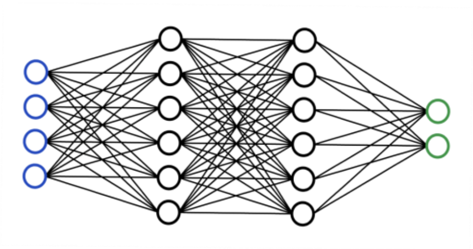
Repeat for every timestep

Nonlinear Functions Make Reachability Difficult

Smooth nonlinearity



Neural network



Adding them naively would make the optimization problem non-convex

And speed + optimality are essential

OVERT.jl

https://github.com/sisl/OVERT.jl

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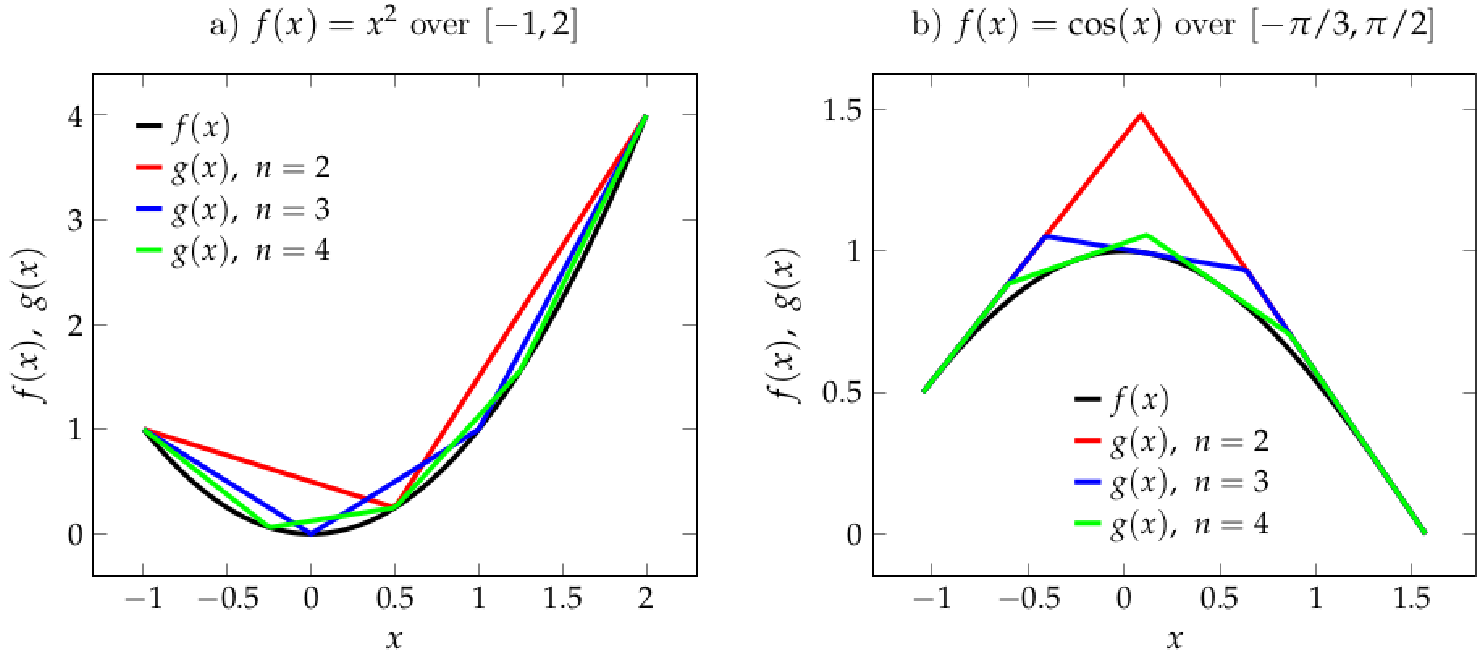
Area Minimal Piecewise Linear Bounds



a) $f(x) = x^2$ over $[-1, 2]$

b) $f(x) = \cos(x)$ over $[-\pi/3, \pi/2]$

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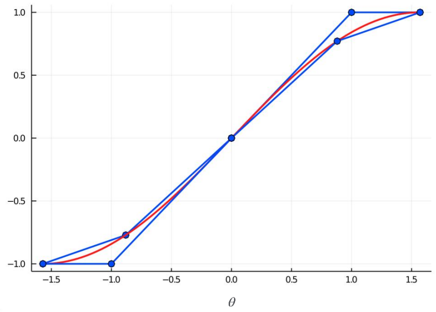
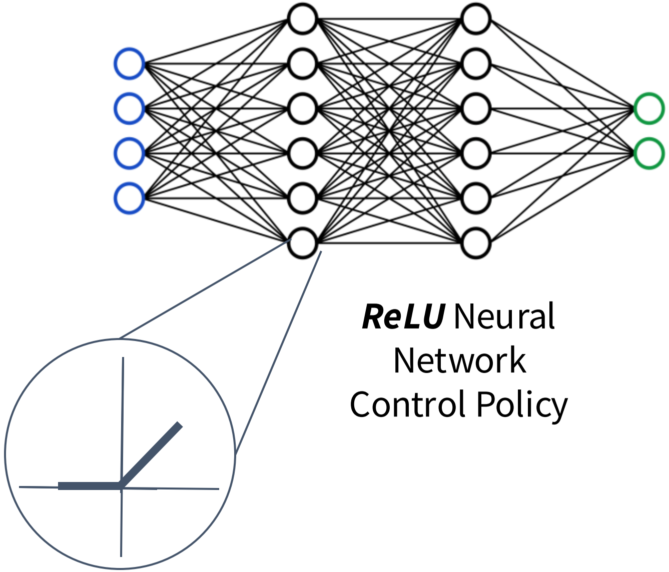
Handling Two Kinds of Nonlinearities



Piecewise linear inclusions of smooth nonlinear functions in dynamics



ReLU Neural Network Control Policy

ReLU Function

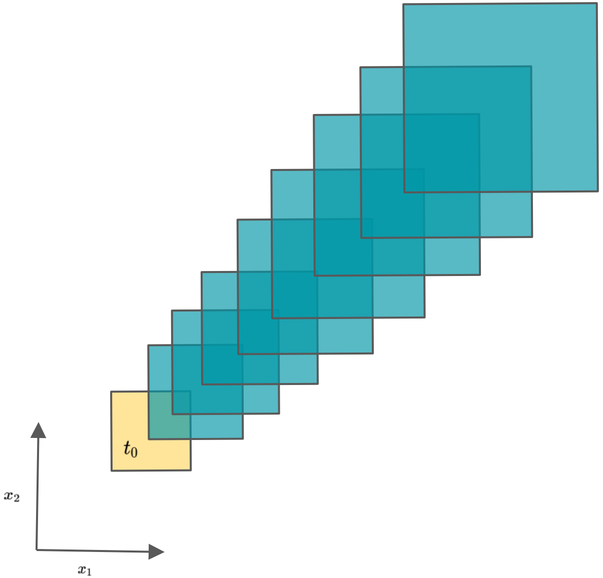
Everything is piecewise-linear!

We can solve an MILP!

Keeping Reachability Tight + Tractable

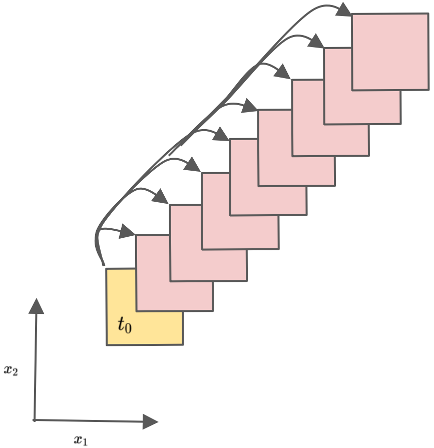
Pure 1-step

Too Conservative



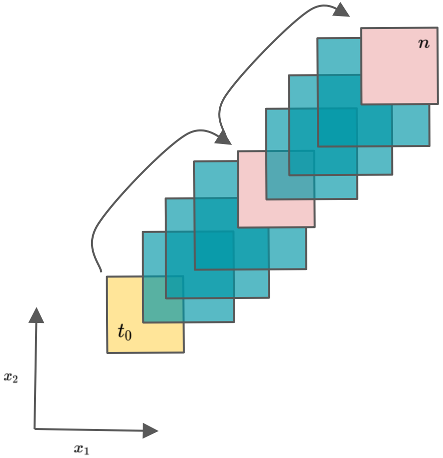
Pure n-step

Intractable



Hand-Tuned Schedule

Impractical

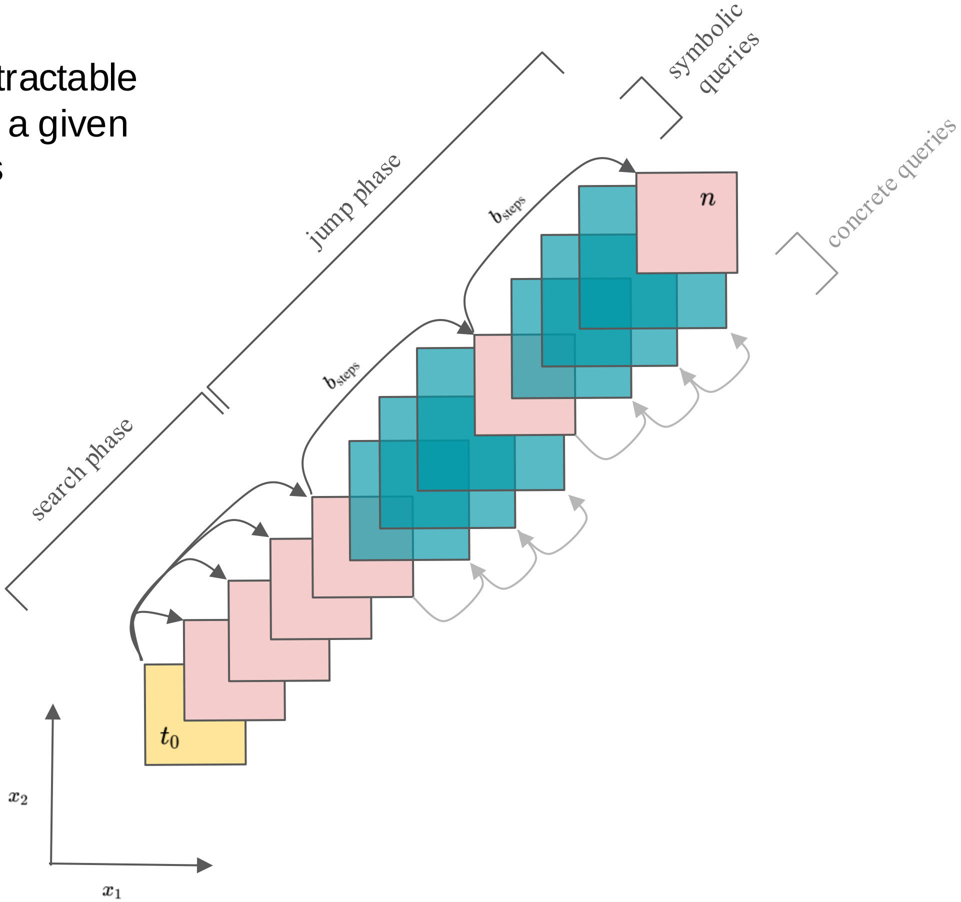


TTT: A **Temporal** Refinement Heuristic for **Tenuously Tractable** Discrete Time Reachability Problems

My newest project

Automatic Hybrid-Symbolic Reachability

Search for the longest tractable n-step query subject to a given time budget in seconds



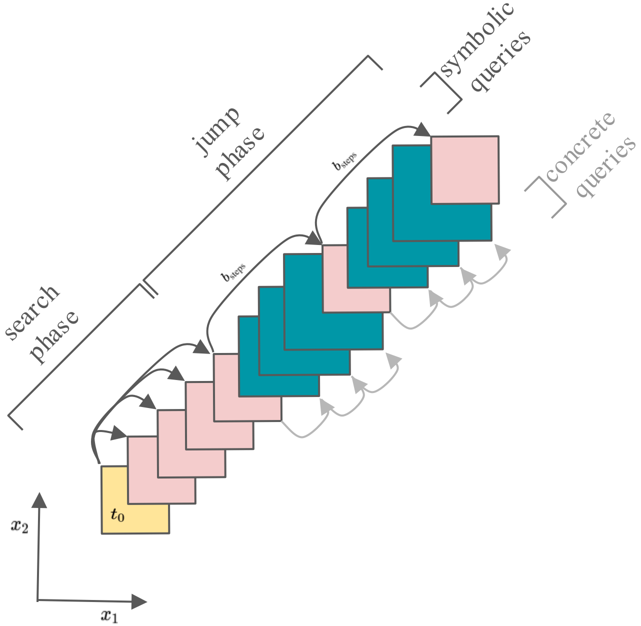
Automatic Hybrid-Symbolic Reachability

- Use a linear estimate of query time as function of number of steps

- At each solve, enforce a time limit through early stopping

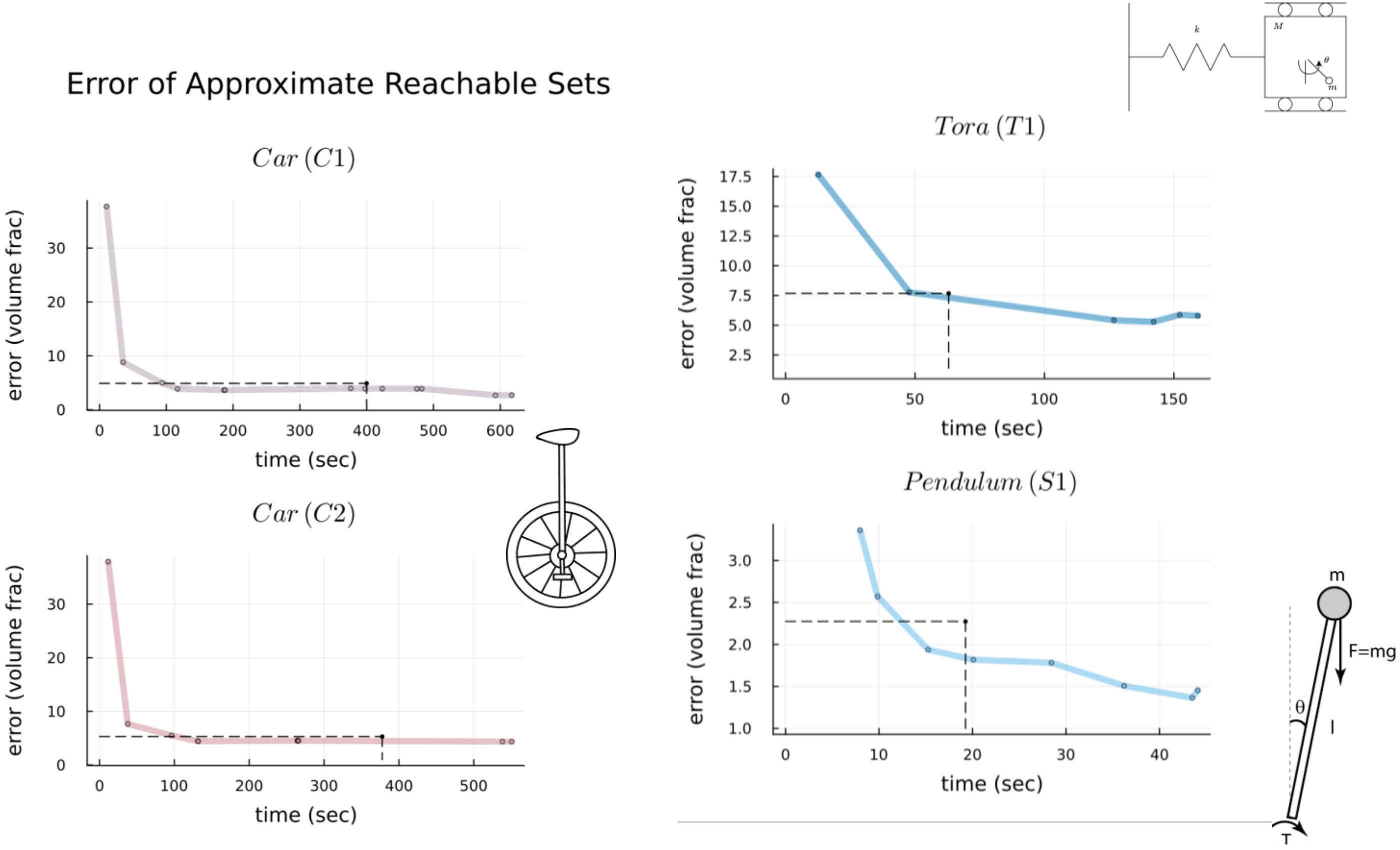
- When using early stopping, must use objective bound to ensure soundness

- If !isfinite(objective_bound(model)) or relative_gap(model) > 0.50, extend the time limit and call optimize!(model) again



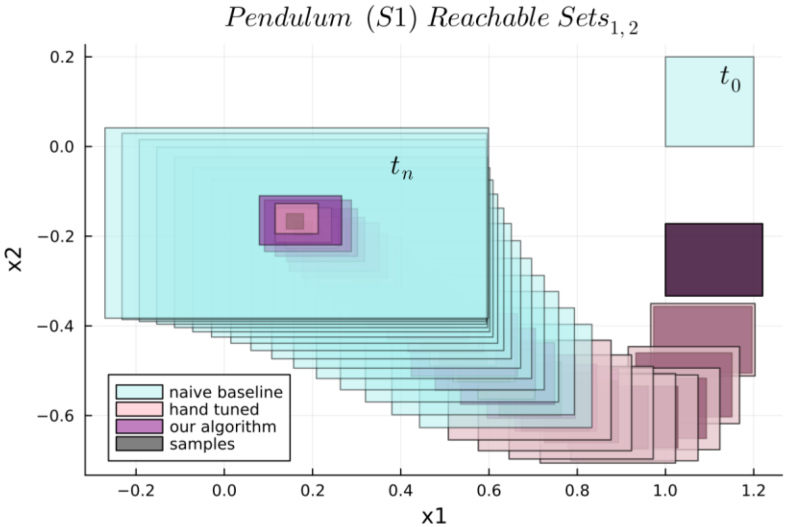
Results

- Can produce reachable sets of varying fidelity given varying time
  - *No more hand-tuning needed!*
- For similar amounts of error as a hand-tuned approach, we are 20-70% faster

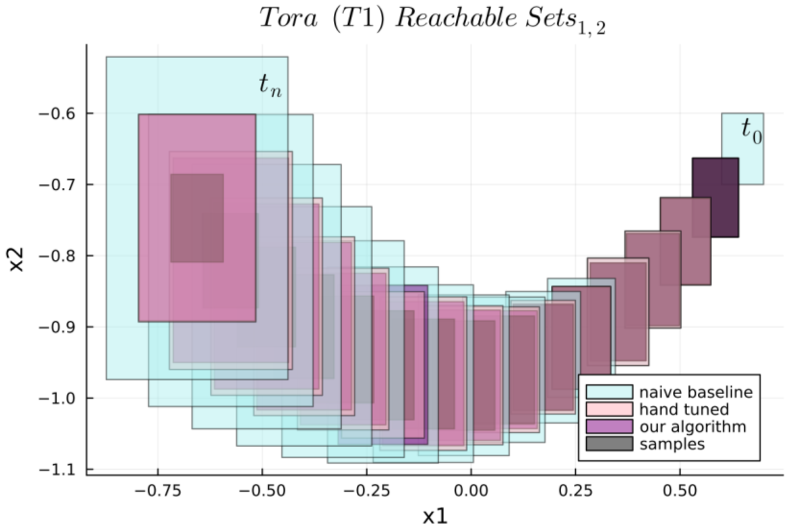


Error of Approximate Reachable Sets

Some Reachable Sets



	Naive	Hand-Tuned	Ours
Error (volume)	11.9	2.27	1.94
Error (radius)	3.79	1.61	1.59
Time	7.84s	19.3s	**15.3s**



	Naive	Hand-Tuned	Ours
Error (volume)	31.7	7.67	7.78
Error (radius)	1.81	1.41	1.38
Time	6.62s	63.0s	**47.8s**

Julia Package Summary

My Packages Discussed in this Talk

OVERT.jl https://github.com/sisl/OVERT.jl

Overapproximations of nonlinear functions

OVERTVerify.jl https://github.com/sisl/OVERTVerify.jl ✸

Reachability analysis for nonlinear dynamical systems with neural network control policies

AutomaticRefinement.jl (potentially coming soon to a GitHub near you)

The automatic temporal refinement work discussed here

Other Useful Related Packages

Expr2MIP.jl https://github.com/chelseas/Expr2MIP.jl ✸

Encode arbitrary expressions of type Expr *into JuMP MILP models (depends on OVERT.jl for smooth nonlinear functions)*
(My package)

NeuralVerification.jl https://github.com/sisl/NeuralVerification.jl ✸

Pedagogical implementations of various neural network verification algorithms (Written by collaborators)

(✸ == *depends on* JuMP.jl)

Info & Links

Chelsea Sidrane, PhD

chelse@kth.se

website:

Thanks for listening!

Reach out if you want to discuss :)

Reserve Slides

OVERT.jl

OVERT’s Overapproximation

$$\dot{\theta}_{t+1} = \dot{\theta}_t + c_1 \sin(\theta_t) + c_2\ u_t$$

1) Re-write nonlinear multi-dimensional functions as one-dimensional or affine functions

$$v_1 = \sin(\theta_t)$$

$$\dot{\theta}_{t+1} = \dot{\theta}_t + c_1\ v_1 + c_2\ u_t$$

Made much easier by Julia’s Expr type and easy symbolic manipulation

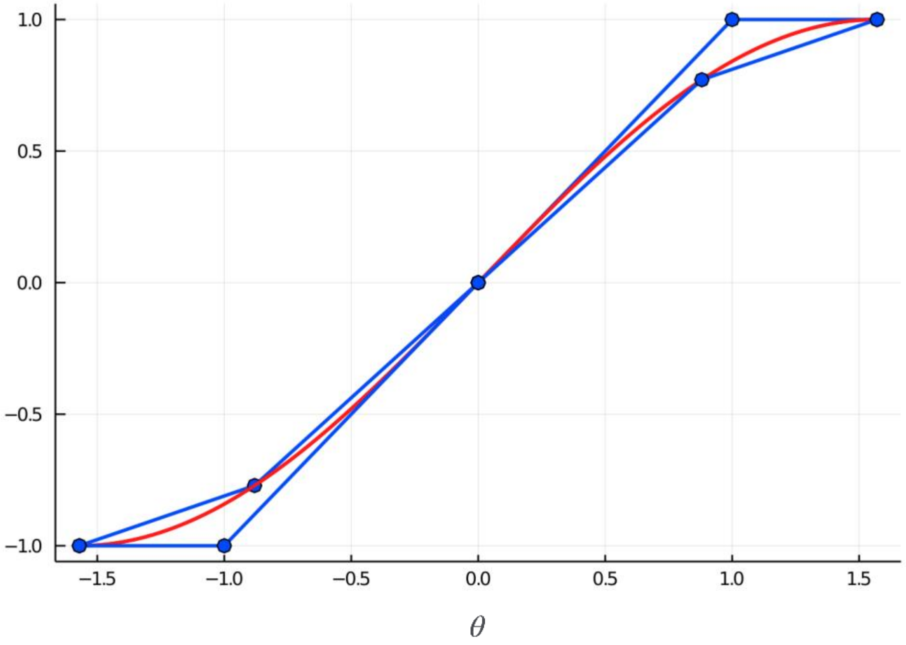
1) Overapproximate each nonlinear one-dimensional function

$$\sin_{LB}(\theta_t) \leq v_1 \leq \sin_{UB}(\theta_t)$$

Area Minimal Piecewise Linear Bounds

$\sin$

$\sin_{LB}, \ \sin_{UB}$



Implementation of OVERT.jl Minimum Area Bounds

solve system of equations = 0 to optimize bound points x_i using NLsolve.jl ⍰

Optimality not needed but always a perk

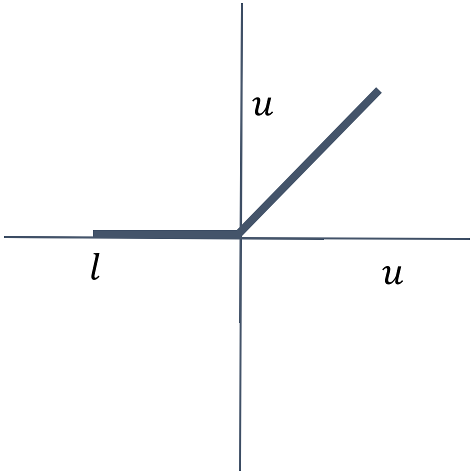
$$
\begin{aligned}
x_0 &= a \\
x_1 &= h\left(a, \frac{x_1 + x_2}{2}\right) \\
x_i &= h\left(\frac{x_{i-1} + x_i}{2}, \frac{x_i + x_{i+1}}{2}\right), \quad i \in \{2, \cdots, n-2\} \\
x_{n-1} &= h\left(\frac{x_{n-1} + x_n}{2}, b\right) \\
x_n &= b
\end{aligned}
$$

OVERTVerify.jl

Encoding the Problem in an MILP

- All piecewise linear functions can be written in terms of max and min
- Max can be encoded as shown using a unique upper and lower bound for each instance instead of `big-M`

Constraints



$$z = max(\hat{z}, 0)$$

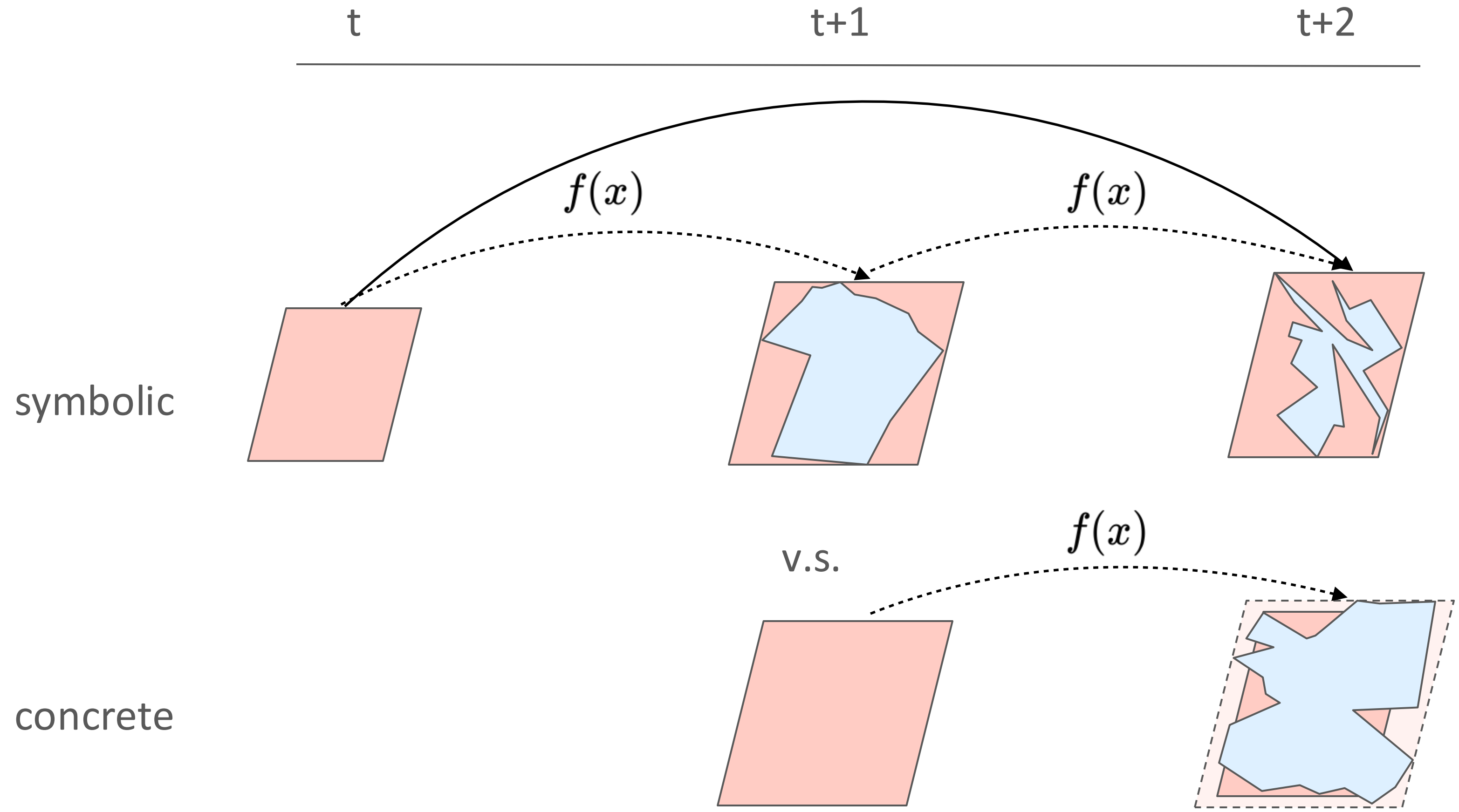
$$\boxed{\begin{aligned} z &\geq \hat{z} \\ z &\geq 0 \\ z &\leq \hat{z} - l(1-\delta) \\ z &\leq u\delta \\ \text{for } \hat{z} &\in [\mathrm{l}, \mathrm{u}] \end{aligned}}$$

Tjeng, V., Xiao, K. and Tedrake, R., 2017. Evaluating robustness of neural networks with mixed integer programming. *arXiv preprint arXiv:1711.07356*.

How to Get Bounds?

- Dynamics with Smooth Nonlinearities
  - From OVERT.jl which uses interval arithmetic, and from reach sets
- Neural Network
  - Using MaxSens
    - [W. Xiang, H. Tran, and T. T. Johnson, “Output reachable set estimation and verification for multilayer neural networks,” pp. 5777–5783, IEEE Transactions on Neural Networks and Learning Systems, vol. 29, no. 11, Nov. 2018.]

Concrete Vs. Symbolic Reachability


t
t+1
t+2
$f(x)$
$f(x)$
symbolic
v.s.
$f(x)$
concrete